

# EIGENVALUES OF THE BUCKLING PROBLEM OF ARBITRARY ORDER ON BOUNDED DOMAINS OF $\mathbb{M} \times \mathbb{R}$

QIAOLING WANG AND CHANGYU XIA

**ABSTRACT.** We obtain universal inequalities for eigenvalues of the buckling problem of arbitrary order on bounded domains in  $\mathbb{M} \times \mathbb{R}$ .

## 1. INTRODUCTION

Let  $\Omega$  be a bounded domain with smooth boundary in an  $n(\geq 2)$ -dimensional Riemannian manifold  $\mathbb{N}$  and denote by  $\Delta$  the Laplace operator acting on functions on  $\mathbb{N}$ . Let  $\nu$  be the outward unit normal vector field of  $\partial\Omega$  and let us consider the following eigenvalue problem :

$$(1.1) \quad \begin{cases} (-\Delta)^l u = -\Lambda \Delta u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0, & \text{on } \partial\Omega, \end{cases}$$

where  $l$  is an integer no less than 2. This problem is called the buckling problem of order  $l$  which has interpretations in physics, that is, it describes the critical buckling load of a clamped plate subjected to a uniform compressive force around its boundary. Let

$$0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots$$

denote the successive eigenvalues for (1.1). Here each eigenvalue is repeated according to its multiplicity. An important theme of geometric analysis is to estimate these (and other) eigenvalues. When  $l = 2$  and  $\Omega$  is a bounded domain in an  $n$ -dimensional Euclidean space  $\mathbb{R}^n$ , in answering a long standing question of Payne-Pólya-Weinberg [8], Cheng and Yang [2] have obtained the following universal inequality:

$$(1.2) \quad \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n+2)}{n^2} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i.$$

On the other hand, when  $\Omega$  is a bounded domain in an  $n$ -dimensional unit sphere  $\mathbb{S}^n$  and  $l = 2$ , the following universal inequality has been proved in [9]:

$$(1.3) \quad \begin{aligned} & 2 \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \\ & \leq \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \left( \delta \Lambda_i + \frac{\delta^2 (\Lambda_i - (n-2))}{4(\delta \Lambda_i + n-2)} \right) + \frac{1}{\delta} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \left( \Lambda_i + \frac{(n-2)^2}{4} \right), \end{aligned}$$

where  $\delta$  is an arbitrary positive constant.

---

2000 *Mathematics Subject Classification* : 35P15, 53C20, 53C42, 58G25

Key words and phrases: Universal inequality for eigenvalues, the buckling problem of arbitrary order,  $\mathbb{M} \times \mathbb{R}$ .

This work was partially supported by CNPq and PROCAD/CAPEs.

Recently, Cheng-Yang [3] improved (1.2) to

$$(1.4) \quad \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4(n + \frac{4}{3})}{n^2} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i$$

and conjectured that the following inequality is true:

$$(1.5) \quad \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i.$$

The inequality (1.3) has been improved in [6] and [3]. For arbitrary  $l$ , when  $\Omega$  is a bounded domain in a Euclidean space  $\mathbb{R}^n$  or a unit sphere, Jost, Jost-Li, Wang and Xia [6] proved some universal inequalities which have been improved by Cheng, Qi, Wang and Xia [1]. For eigenvalues of the problem (1.1), in addition to considering the possible sharp inequalities, another interesting question is to know for what kind of complete manifolds there exists universal upper bound on  $\Lambda_{k+1}$  in terms of  $\Lambda_1, \dots, \Lambda_k$  independent of the domains. Recently, universal inequalities for eigenvalues of the (1.1) have been obtained for bounded domains of some special Ricci flat manifolds in [4].

In this paper, we prove the following result.

**Theorem 1.1.** *Let  $\mathbb{M}$  be a complete Riemannian manifold and let  $\Omega$  be a bounded domain of the Riemannian product manifold  $\mathbb{M} \times \mathbb{R}$ . Denote by  $\Lambda_i$  the  $i$ th eigenvalue of the problem (1.1). Then for any positive non-increasing monotone sequence  $\{\delta_i\}_{i=1}^k$  we have*

$$(1.6) \quad \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq \left( 2l^2 - \frac{11}{3}l + \frac{5}{3} \right) \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{(l-2)/(l-1)} + \sum_{i=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)}.$$

**Corollary 1.2.** *Under the same conditions as in Theorem 1.1, we have*

$$(1.7) \quad \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \leq 4 \left( 2l^2 - \frac{11}{3}l + \frac{5}{3} \right) \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i.$$

*Proof of Corollary 1.2.* Taking

$$\delta_1 = \dots = \delta_k = \frac{\left( \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)} \right)^{\frac{1}{2}}}{\left( \left( 2l^2 - \frac{11}{3}l + \frac{5}{3} \right) \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{(l-2)/(l-1)} \right)^{\frac{1}{2}}}$$

in (1.6) and using (Cf. [7])

$$(1.8) \quad \begin{aligned} & \left( \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \Lambda_i^{(l-2)/(l-1)} \right) \left( \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)} \right) \\ & \leq \left( \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \right) \left( \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i) \Lambda_i \right), \end{aligned}$$

we get (1.7).

## 2. A PROOF OF THEOREM 1.1

Before proving our result, let us recall some known facts we need. Let  $\mathbb{M}$  be a complete manifold and let  $\mathbb{N} = \mathbb{M} \times \mathbb{R} = \{(x, t) | x \in \mathbb{M}, t \in \mathbb{R}\}$  with the product metric of  $\mathbb{M}$  and  $\mathbb{R}$ . Let  $\Omega(\subset \mathbb{N})$  be a bounded domain with smooth boundary. Since any complete Riemannian manifold can be isometrically embedded in some Euclidean space, we can think of our  $\mathbb{N}$  as a submanifold of some  $\mathbb{R}^q$ . Let us denote by  $\langle \cdot, \cdot \rangle$  the canonical metric on  $\mathbb{R}^q$  as well as that induced on  $\mathbb{N}$ . Denote by  $\Delta$  and  $\nabla$  the Laplacian and the gradient operator of  $\mathbb{N}$ , respectively. Let  $u_i$  be the  $i$ -th orthonormal eigenfunction of the problem (1.1) corresponding to the eigenvalue  $\Lambda_i$ ,  $i = 1, 2, \dots$ , that is,

$$(2.1) \quad \begin{cases} (-\Delta)^l u_i = -\Lambda_i \Delta u_i, & \text{in } \Omega, \\ u_i = \frac{\partial u_i}{\partial \nu} = \dots = \frac{\partial^{l-1} u_i}{\partial \nu^{l-1}} = 0, & \text{on } \partial\Omega, \\ (u_i, u_j)_D = \int_{\Omega} \langle \nabla u_i, \nabla u_j \rangle = \delta_{ij}, & \forall i, j. \end{cases}$$

For functions  $f$  and  $g$  on  $\Omega$ , the *Dirichlet inner product*  $(f, g)_D$  of  $f$  and  $g$  is given by

$$(f, g)_D = \int_{\Omega} \langle \nabla f, \nabla g \rangle.$$

The Dirichlet norm of a function  $f$  is defined by

$$\|f\|_D = \{(f, f)_D\}^{1/2} = \left( \int_{\Omega} |\nabla f|^2 \right)^{1/2}.$$

For each  $k = 1, \dots, l$ , let  $\nabla^k$  denote the  $k$ -th covariant derivative operator on  $\mathbb{N}$ , defined in the usual weak sense. For a function  $f$  on  $\Omega$ , the squared norm of  $\nabla^k f$  is defined as (cf. [5])

$$(2.2) \quad |\nabla^k f|^2 = \sum_{i_1, \dots, i_k=1}^n \left( \nabla^k f(e_{i_1}, \dots, e_{i_k}) \right)^2,$$

where  $e_1, \dots, e_n$  are orthonormal vector fields locally defined on  $\Omega$ . Define the Sobolev space  $H_l^2(\Omega)$  by

$$H_l^2(\Omega) = \{f : f, |\nabla f|, \dots, |\nabla^l f| \in L^2(\Omega)\}.$$

Then  $H_l^2(\Omega)$  is a Hilbert space with respect to the inner product  $\langle \cdot, \cdot \rangle$ :

$$(2.3) \quad \langle \langle f, g \rangle \rangle = \int_{\Omega} \left( \sum_{k=0}^l \nabla^k f \cdot \nabla^k g \right),$$

where

$$\nabla^k f \cdot \nabla^k g = \sum_{i_1, \dots, i_k=1}^n \nabla^k f(e_{i_1}, \dots, e_{i_k}) \nabla^k g(e_{i_1}, \dots, e_{i_k}).$$

Consider the subspace  $H_{l,D}^2(\Omega)$  of  $H_l^2(\Omega)$  defined by

$$H_{l,D}^2(\Omega) = \left\{ f \in H_l^2(\Omega) : f|_{\partial\Omega} = \frac{\partial f}{\partial \nu} \Big|_{\partial\Omega} = \dots = \frac{\partial^{l-1} f}{\partial \nu^{l-1}} \Big|_{\partial\Omega} = 0 \right\}.$$

The operator  $(-\Delta)^l$  defines a self-adjoint operator acting on  $H_{l,D}^2(\Omega)$  with discrete eigenvalues  $0 < \Lambda_1 \leq \dots \leq \Lambda_k \leq \dots$  for the buckling problem (1.1) and the eigenfunctions  $\{u_i\}_{i=1}^{\infty}$  defined in

(2.1) form a complete orthonormal basis for the Hilbert space  $H_{l,D}^2(\Omega)$ . If  $\phi \in H_{l,D}^2(\Omega)$  satisfies  $(\phi, u_j)_D = 0$ ,  $\forall j = 1, 2, \dots, k$ , then the Rayleigh-Ritz inequality tells us that

$$(2.4) \quad \Lambda_{k+1} \|\phi\|_D^2 \leq \int_{\Omega} \phi(-\Delta)^l \phi.$$

For vector-valued functions  $F = (f_1, f_2, \dots, f_m)$ ,  $G = (g_1, g_2, \dots, g_m) : \Omega \rightarrow \mathbb{R}^q$ , we define an inner product  $(F, G)$  by

$$(F, G) \equiv \int_{\Omega} \langle F, G \rangle = \int_{\Omega} \sum_{\alpha=1}^q f_{\alpha} g_{\alpha}.$$

The norm of  $F$  is given by

$$\|F\| = (F, F)^{1/2} = \left\{ \int_{\Omega} \sum_{\alpha=1}^q f_{\alpha}^2 \right\}^{1/2}.$$

Let  $\mathbf{H}_1^2(\Omega)$  be the Hilbert space of vector-valued functions given by

$$\mathbf{H}_1^2(\Omega) = \{F = (f_1, \dots, f_q) : \Omega \rightarrow \mathbb{R}^q; f_{\alpha}, |\nabla f_{\alpha}| \in L^2(\Omega), \text{ for } \alpha = 1, \dots, m\}$$

with inner product  $\langle, \rangle_1$ :

$$\langle F, G \rangle_1 = (F, G) + \int_{\Omega} \sum_{\alpha=1}^q \langle \nabla f_{\alpha}, \nabla g_{\alpha} \rangle.$$

Observe that a vector field on  $\Omega$  can be regarded as a vector-valued function from  $\Omega$  to  $\mathbb{R}^q$ . Let  $\mathbf{H}_{1,D}^2(\Omega) \subset \mathbf{H}_1^2(\Omega)$  be a subspace of  $\mathbf{H}_1^2(\Omega)$  spanned by the vector-valued functions  $\{\nabla u_i\}_{i=1}^{\infty}$ , which form a complete orthonormal basis of  $\mathbf{H}_{1,D}^2(\Omega)$ . For any  $f \in H_{l,D}^2(\Omega)$ , we have  $\nabla f \in \mathbf{H}_{1,D}^2(\Omega)$  and for any  $X \in \mathbf{H}_{1,D}^2(\Omega)$ , there exists a function  $f \in H_{l,D}^2(\Omega)$  such that  $X = \nabla f$ .

Consider the function  $g : \Omega \rightarrow \mathbb{R}$  given by  $g(x, t) = t$ . The vector fields  $g \nabla u_i$  can be decomposed as

$$(2.5) \quad g \nabla u_i = \nabla h_i + \mathbf{W}_i,$$

where  $h_i \in H_{l,D}^2(\Omega)$ ,  $\nabla h_i$  is the projection of  $g \nabla u_i$  in  $\mathbf{H}_{1,D}^2(\Omega)$ ,  $\mathbf{W}_i \perp \mathbf{H}_{1,D}^2(\Omega)$  and

$$(2.6) \quad \begin{cases} \mathbf{W}_i|_{\partial\Omega} = 0, \\ \operatorname{div} \mathbf{W}_i = 0. \end{cases}$$

Since  $\mathbb{N}$  is endowed with the product metric of  $\mathbb{M}$  and  $\mathbb{R}$ , it is easy to see that

$$(2.7) \quad |\nabla g| = 1, \quad \nabla^2 g = 0, \quad \operatorname{Ric}(\nabla g, X) = 0,$$

for any vector fields  $X \in \mathfrak{X}(\Omega)$ , where  $\operatorname{Ric}$  denotes the Ricci tensor of  $\mathbb{N}$ . Substituting (2.7) into the Bochner formula, we obtain for any smooth  $f : \Omega \rightarrow \mathbb{R}$  that

$$(2.8) \quad \begin{aligned} \Delta \langle \nabla f, \nabla g \rangle &= 2 \nabla^2 f \cdot \nabla^2 g + \langle \nabla f, \nabla(\Delta g) \rangle + \langle \nabla g, \nabla(\Delta f) \rangle + 2 \operatorname{Ric}(\nabla g, \nabla f) \\ &= \langle \nabla g, \nabla(\Delta f) \rangle. \end{aligned}$$

In the proof of Theorem 1.1, we shall use (2.5)-(2.8) repeatedly. Now we can prove the main result in this paper.

**Proof of Theorem 1.1.** For each  $i = 1, \dots, k$ , let us consider the functions  $\phi_i : \Omega \rightarrow \mathbb{R}$  given by

$$(2.9) \quad \phi_i = h_i - \sum_{j=1}^k b_{ij} u_j,$$

where  $b_{ij} = \int_{\Omega} g \langle \nabla u_i, \nabla u_j \rangle = b_{ji}$  and  $h_i$  is given in (2.5). Since

$$\phi_i|_{\partial\Omega} = \frac{\partial\phi_i}{\partial\nu}\Big|_{\partial\Omega} = \cdots = \frac{\partial^{l-1}\phi_i}{\partial\nu^{l-1}}\Big|_{\partial\Omega} = 0, \int_{\Omega} \langle \nabla\phi_i, \nabla u_j \rangle = 0, \quad j = 1, \dots, k,$$

we have from the Rayleigh-Ritz inequality that

$$(2.10) \quad \Lambda_{k+1} \int_{\Omega} |\nabla\phi_i|^2 \leq \int_{\Omega} \phi_i (-\Delta)^l \phi_i, \quad \forall i = 1, \dots, k.$$

Using (2.5)-(2.9) and the same calculations as in [4] we get (Cf. (3.17)-(3.20) in [4])

$$(2.11) \quad \begin{aligned} & \int_{\Omega} \phi_i (-\Delta)^l \phi_i \\ &= \int_{\Omega} (-1)^l (2l^2 - 4l + 3) \langle \nabla g, \nabla u_i \rangle \langle \nabla g, \nabla (\Delta^{l-2} u_i) \rangle \\ &+ \int_{\Omega} (-1)^l (-l + 1) u_i \Delta^{l-1} u_i + \Lambda_i \left\{ \int_{\Omega} g^2 |\nabla u_i|^2 - \int_{\Omega} u_i^2 \right\} - \sum_{j=1}^k \Lambda_j b_{ij}^2, \end{aligned}$$

$$(2.12) \quad 2 \int_{\Omega} \langle g \nabla u_i, \nabla \langle \nabla g, \nabla u_i \rangle \rangle = -1,$$

and

$$(2.13) \quad c_{ij} \equiv \int_{\Omega} \langle \nabla \langle \nabla g, \nabla u_i \rangle, \nabla u_j \rangle = -c_{ji}.$$

Substituting

$$(2.14) \quad \|g \nabla u_i\|^2 = \|\nabla h_i\|^2 + \|\mathbf{W}_i\|^2, \quad \|\nabla h_i\|^2 = \|\nabla \phi_i\|^2 + \sum_{j=1}^k b_{ij}^2$$

and (2.12) into (2.11), we get

$$(2.15) \quad \begin{aligned} & (\Lambda_{k+1} - \Lambda_i) \|\nabla \phi_i\|^2 \\ & \leq \int_{\Omega} (-1)^l (2l^2 - 4l + 3) \langle \nabla g, \nabla u_i \rangle \langle \nabla g, \nabla (\Delta^{l-2} u_i) \rangle \\ & + \int_{\Omega} (-1)^l (-l + 1) u_i \Delta^{l-1} u_i + \Lambda_i (\|u_i\|^2 - \|\mathbf{W}_i\|^2) + \sum_{j=1}^k (\Lambda_i - \Lambda_j) b_{ij}^2, \end{aligned}$$

where for a function  $f : \Omega \rightarrow \mathbb{R}$ ,  $\|f\|^2 = \int_{\Omega} f^2$  and for a vector field  $X \in \mathfrak{X}(\Omega)$ ,  $\|X\|^2 = \int_{\Omega} |X|$ ,  $|X|$  being the length of  $X$ .

It is easy to see from (2.5) and (2.9) that (Cf. (3.20) in [4])

$$(2.16) \quad 1 + 2 \sum_{j=1}^k b_{ij} c_{ij} = -2 \int_{\Omega} \langle \nabla \phi_i, \nabla \langle \nabla g, \nabla u_i \rangle \rangle.$$

Thus, we have

$$\begin{aligned}
(2.17) \quad & (\Lambda_{k+1} - \Lambda_i)^2 \left( 1 + 2 \sum_{j=1}^k b_{ij} c_{ij} \right) \\
&= (\Lambda_{k+1} - \Lambda_i)^2 \left( -2 \int_{\Omega} \left\langle \nabla \phi_i, \nabla \langle \nabla g, \nabla u_i - \sum_{j=1}^k c_{ij} \nabla u_j \rangle \right\rangle \right) \\
&\leq \delta_i (\Lambda_{k+1} - \Lambda_i)^3 \|\nabla \phi_i\|^2 + \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \left( \|\nabla \langle \nabla g, \nabla u_i \rangle\|^2 - \sum_{j=1}^k c_{ij}^2 \right).
\end{aligned}$$

Summing on  $i$  from 1 to  $k$  in (2.17), using (2.15) and noticing  $a_{ij} = a_{ji}$ ,  $c_{ij} = -c_{ji}$ , we get

$$\begin{aligned}
(2.18) \quad & \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 - 2 \sum_{i,j=1}^k (\Lambda_{k+1} - \Lambda_i)(\Lambda_i - \Lambda_j) b_{ij} c_{ij} \\
&\leq \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left( \int_{\Omega} (-1)^l (2l^2 - 4l + 3) \langle \nabla g, \nabla u_i \rangle \langle \nabla g, \nabla (\Delta^{l-2} u_i) \rangle \right. \\
&\quad \left. + \int_{\Omega} (-1)^l (-l + 1) u_i \Delta^{l-1} u_i + \Lambda_i (\|u_i\|^2 - \|\mathbf{W}_i\|^2) \right) + \sum_{i=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \|\nabla \langle \nabla g, \nabla u_i \rangle\|^2 \\
&\quad + \sum_{i,j=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 (\Lambda_i - \Lambda_j) b_{ij}^2 - \sum_{i,j=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) c_{ij}^2.
\end{aligned}$$

Since  $\{\delta_i\}$  is a non-increasing monotone sequence, we have

$$\begin{aligned}
(2.19) \quad & \sum_{i,j=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 (\Lambda_i - \Lambda_j) b_{ij}^2 \\
&= - \sum_{i,j=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i) (\Lambda_i - \Lambda_j)^2 b_{ij}^2 + \frac{1}{2} \sum_{i,j=1}^k (\Lambda_{k+1} - \Lambda_i) (\Lambda_{k+1} - \Lambda_j) (\delta_i - \delta_j) (\Lambda_i - \Lambda_j) b_{ij}^2 \\
&\leq - \sum_{i,j=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i) (\Lambda_i - \Lambda_j)^2 b_{ij}^2.
\end{aligned}$$

Substituting (2.19) into (2.18), we conclude

$$\begin{aligned}
(2.20) \quad & \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \\
&\leq \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left( \int_{\Omega} (-1)^l (2l^2 - 4l + 3) \langle \nabla g, \nabla u_i \rangle \langle \nabla g, \nabla (\Delta^{l-2} u_i) \rangle \right. \\
&\quad \left. + \int_{\Omega} (-1)^l (-l + 1) u_i \Delta^{l-1} u_i - \Lambda_i (\|u_i\|^2 - \|\mathbf{W}_i\|^2) \right) + \sum_{i=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \|\nabla \langle \nabla g, \nabla u_i \rangle\|^2.
\end{aligned}$$

Before we can finish the proof of Theorem 1.1, we shall need some lemmas.

**Lemma 2.1.** *We have*

i) (Cf. (2.6) in [6]).  $0 \leq \int_{\Omega} u_i (-\Delta)^k u_i \leq \Lambda_i^{(k-1)/(l-1)}$ ,  $k = 1, \dots, l-1$ .

ii) (Cf. (3.26) in [4]).  $\|\nabla \langle \nabla g, \nabla u_i \rangle\|^2 \leq \Lambda_i^{1/(l-1)}$ .

iii) (Cf. (3.33) in [4]).  $\int_{\Omega} \langle \nabla g, \nabla ((-\Delta)^{l-2} u_i) \rangle \langle \nabla g, \nabla u_i \rangle \leq \Lambda_i^{(l-2)/(l-1)}$ .

**Lemma 2.2.** *We have*

$$(2.21) \quad -\Lambda_i(\|u_i\|^2 - \|\mathbf{W}_i\|^2) \leq -\frac{2(l+1)}{3} \int_{\Omega} \langle \nabla g, \nabla ((-\Delta)^{l-2} u_i) \rangle \langle \nabla g, \nabla u_i \rangle \\ + \frac{1}{3} \int_{\Omega} u_i (-\Delta)^{l-1} u_i.$$

*Proof of Lemma 2.2.* Firstly, we prove the following equality:

$$(2.22) \quad -\Lambda_i(\|u_i\|^2 - \|\mathbf{W}_i\|^2) = -\frac{l+1}{2} \int_{\Omega} \langle \nabla g, \nabla ((-\Delta)^{l-2} u_i) \rangle \langle \nabla g, \nabla u_i \rangle \\ - \frac{1}{4} \Lambda_i \|u_i\|^2 + \frac{1}{4} \int_{\Omega} u_i (-\Delta)^{l-1} u_i.$$

Observe that  $\nabla(gu_i) = u_i \nabla g + g \nabla u_i \in \mathbf{H}_{1,D}^2(\Omega)$ . Set  $y_i = gu_i - h_i$ ; then

$$(2.23) \quad u_i \nabla g = \nabla y_i - \mathbf{W}_i.$$

and so

$$(2.24) \quad \|u_i\|^2 = \|u_i \nabla g\|^2 = \|\mathbf{W}_i\|^2 + \|\nabla y_i\|^2.$$

We have

$$(2.25) \quad -\int_{\Omega} y_i g \Delta u_i = \int_{\Omega} \langle \nabla y_i, g \nabla u_i \rangle + \int_{\Omega} y_i \langle \nabla g, \nabla u_i \rangle \\ = \int_{\Omega} \langle \nabla y_i, g \nabla u_i \rangle - \int_{\Omega} \langle \nabla y_i, u_i \nabla g \rangle \\ = \int_{\Omega} \langle \nabla y_i, g \nabla u_i \rangle - \|\nabla y_i\|^2 \\ = \int_{\Omega} \langle u_i \nabla g + \mathbf{W}_i, g \nabla u_i \rangle - \|\nabla y_i\|^2 \\ = \frac{1}{4} \int_{\Omega} \langle \nabla g^2, \nabla u_i^2 \rangle + \|\mathbf{W}_i\|^2 - \|\nabla y_i\|^2 \\ = -\frac{1}{4} \int_{\Omega} u_i^2 \Delta g^2 + \|\mathbf{W}_i\|^2 - \|\nabla y_i\|^2 \\ = -\frac{1}{2} \|u_i\|^2 + \|\mathbf{W}_i\|^2 - \|\nabla y_i\|^2 \\ = -2(\|u_i\|^2 - \|\mathbf{W}_i\|^2) + \frac{1}{2} \|u_i\|^2,$$

$$\begin{aligned}
(2.26) \quad & \int_{\Omega} \Delta^{l-1} u_i \langle \nabla y_i, \nabla g \rangle \\
&= - \int_{\Omega} y_i \langle \nabla (\Delta^{l-1} u_i), \nabla g \rangle \\
&= - \int_{\Omega} y_i \Delta \langle \nabla (\Delta^{l-2} u_i), \nabla g \rangle \\
&= \int_{\Omega} \langle \nabla y_i, \nabla \langle \nabla (\Delta^{l-2} u_i), \nabla g \rangle \rangle \\
&= \int_{\Omega} \langle u_i \nabla g, \nabla \langle \nabla (\Delta^{l-2} u_i), \nabla g \rangle \rangle \\
&= - \int_{\Omega} \langle \nabla g, \nabla (\Delta^{l-2} u_i) \rangle \langle \nabla u_i, \nabla g \rangle
\end{aligned}$$

and

$$\begin{aligned}
(2.27) \quad & \int_{\Omega} g u_i \langle \nabla g, \nabla (\Delta^{l-1} u_i) \rangle \\
&= \int_{\Omega} g u_i \Delta^{l-1} \langle \nabla g, \nabla u_i \rangle \\
&= \int_{\Omega} \Delta^{l-1} (g u_i) \langle \nabla g, \nabla u_i \rangle \\
&= - \int_{\Omega} \langle u_i \nabla g, \nabla (\Delta^{l-1} (g u_i)) \rangle \\
&= - \int_{\Omega} \langle \nabla y_i, \nabla (\Delta^{l-1} (g u_i)) \rangle \\
&= \int_{\Omega} y_i \Delta^l (g u_i) \\
&= \int_{\Omega} y_i (2l \langle \nabla (\Delta^{l-1} u_i), \nabla g \rangle + g \Delta^l u_i) \\
&= - 2l \int_{\Omega} \Delta^{l-1} u_i \langle \nabla y_i, \nabla g \rangle + \Lambda_i (-1)^{l-1} \int_{\Omega} y_i g \Delta u_i.
\end{aligned}$$

It follows from (2.25)-(2.27) that

$$\begin{aligned}
(2.28) \quad & \int_{\Omega} g u_i \langle \nabla g, \nabla (\Delta^{l-1} u_i) \rangle \\
&= 2l \int_{\Omega} \langle \nabla g, \nabla (\Delta^{l-2} u_i) \rangle \langle \nabla u_i, \nabla g \rangle + (-1)^{l-1} \Lambda_i \left( -\frac{1}{2} \|u_i\|^2 + 2(\|u_i\|^2 - \|\mathbf{W}_i\|^2) \right).
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
(2.29) \quad & \int_{\Omega} g u_i \langle \nabla g, \nabla (\Delta^{l-1} u_i) \rangle \\
&= \int_{\Omega} g u_i \Delta^{l-1} \langle \nabla u_i, \nabla g \rangle \\
&= \int_{\Omega} \Delta^{l-1} (g u_i) \langle \nabla u_i, \nabla g \rangle \\
&= \int_{\Omega} \left( 2(l-1) \langle \nabla (\Delta^{l-2} u_i), \nabla g \rangle + g \Delta^{l-1} u_i \right) \langle \nabla u_i, \nabla g \rangle
\end{aligned}$$



and

$$(2.30) \quad \int_{\Omega} g u_i \langle \nabla g, \nabla (\Delta^{l-1} u_i) \rangle = - \int_{\Omega} \Delta^{l-1} u_i (u_i + g \langle \nabla u_i, \nabla g \rangle).$$

Adding (2.29) and (2.30), we infer

$$(2.31) \quad \begin{aligned} & \int_{\Omega} g u_i \langle \nabla g, \nabla (\Delta^{l-1} u_i) \rangle \\ &= \int_M \left\{ (l-1) (\langle \nabla (\Delta^{l-2} u_i), \nabla g \rangle \langle \nabla u_i, \nabla g \rangle - \frac{1}{2} u_i \Delta^{l-1} u_i) \right\}. \end{aligned}$$

Combining (2.28) and (2.31), one gets (2.22).

Substituting

$$- \|u_i\|^2 \leq -(\|u_i\|^2 - \|\mathbf{W}_i\|^2)$$

into (2.22), we obtain (2.21). This completes the proof of Lemma 2.2.

Let us continue on the proof of Theorem 1.1. Substituting (2.21) into (2.20) and using Lemma 2.1, we have

$$(2.32) \quad \begin{aligned} & \sum_{i=1}^k (\Lambda_{k+1} - \Lambda_i)^2 \\ & \leq \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left( \left( 2l^2 - 4l + 3 - \frac{2(l+1)}{3} \right) \int_{\Omega} \langle \nabla g, \nabla u_i \rangle \langle \nabla g, \nabla ((-\Delta)^{l-2} u_i) \rangle \right. \\ & \quad \left. + \int_{\Omega} \left( l - 1 + \frac{1}{3} \right) u_i (-\Delta)^{l-1} u_i \right) + \sum_{i=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \|\nabla \langle \nabla g, \nabla u_i \rangle\|^2 \\ & \leq \sum_{i=1}^k \delta_i (\Lambda_{k+1} - \Lambda_i)^2 \left( 2l^2 - \frac{11}{3}l + \frac{5}{3} \right) \Lambda_i^{(l-2)/(l-1)} \\ & \quad + \sum_{i=1}^k \frac{1}{\delta_i} (\Lambda_{k+1} - \Lambda_i) \Lambda_i^{1/(l-1)}. \end{aligned}$$

This completes the proof of Theorem 1.1.

### 3. CONCLUDING REMARKS

By using (2.7), (2.8) and the same arguments as in the proof of Theorem 1.3 in [4], one can prove the following result.

**Theorem 3.1.** *Let  $\mathbb{M}$  be a complete Riemannian manifold and let  $\Omega$  be a bounded domain of the Riemannian product manifold  $\mathbb{M} \times \mathbb{R}$ . Denote by  $\Gamma_i$  be the  $i$ th eigenvalue of the problem*

$$\begin{cases} (-\Delta)^l u = -\Gamma u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}} = 0, & \text{on } \partial \Omega. \end{cases}$$

Then we have

$$\sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i)^2 \leq 2 \left\{ l(2l-1) \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \Gamma_i^{(l-1)/l} \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^k (\Gamma_{k+1} - \Gamma_i) \Gamma_i^{1/l} \right\}^{\frac{1}{2}}.$$

## REFERENCES

- [1] Cheng, Q.M., Qi, X., Wang, Q., Xia, C., Inequalities for eigenvalues of the buckling problem of arbitrary order, arXiv: 1010.2327v1.
- [2] Cheng, Q.M., Yang, H.C., Universal bounds for eigenvalues of a buckling problem, *Comm. Math. Phys.*, **262**(2006), 663-675.
- [3] Cheng, Q.M., Yang, H.C., Universal bounds for eigenvalues of a buckling problem II, *Trans. Amer. Math. Soc.*, **364**(2012), 6139-6158.
- [4] Du, F., Wu, C., Li, G., Xia, C., Eigenvalues of the buckling problem of arbitrary order and of the polyharmonic operator on Ricci flat manifolds, *J. Math. Anal. Appl.* **417**(2014), 601-621.
- [5] Hebey, H., *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*. Courant Lecture Notes in Mathematics, 5. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999. x+309 pp.
- [6] Jost, J., Li-Jost, X., Wang, Q., Xia, C., Universal inequalities for eigenvalues of the buckling problem of arbitrary order, *Comm. Part. Diff. Equ.* **35**(2010), 1563-1589.
- [7] Jost, J., Li-Jost, X., Wang, Q., Xia, C., Universal bounds for eigenvalues of polyharmonic operator, *Trans. Am. Math. Soc.*, **363**(4)(2011), 1821-1854.
- [8] Payne, L.E., Pólya, G., Weinberger, H.F., On the ratio of consecutive eigenvalues, *J. Math. Phys.*, **35**(1956), 289-298.
- [9] Wang, Q., Xia, C., Universal bounds for eigenvalues of the buckling problem on spherical domains, *Comm. Math. Phys.* **270**(2007), 759-775.

QIAOLING WANG,

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE BRASÍLIA, 70910-900-BRASÍLIA-DF, BRAZIL, WANG@MAT.UNB.BR

CHANGYU XIA,

DEPARTAMENTO DE MATEMÁTICA, UNIVERSIDADE DE BRASÍLIA, 70910-900 BRASÍLIA-DF, BRAZIL. XIA@PQ.CNPQ.BR